

Diameter preserving surjections in the geometry of matrices

Wen-ling Huang* Hans Havlicek†

Abstract

We consider a class of graphs subject to certain restrictions, including the finiteness of diameters. Any surjective mapping $\varphi : \Gamma \rightarrow \Gamma'$ between graphs from this class is shown to be an isomorphism provided that the following holds: Any two points of Γ are at a distance equal to the diameter of Γ if, and only if, their images are at a distance equal to the diameter of Γ' . This result is then applied to the graphs arising from the adjacency relations of spaces of rectangular matrices, spaces of Hermitian matrices, and Grassmann spaces (projective spaces of rectangular matrices).

Keywords. Adjacency preserving mapping, diameter preserving mapping, geometry of matrices, Grassmann space.

MSC: 51A50, 15A57.

1 Introduction

Related to his study of analytic functions of several complex variables, L. K. Hua initiated the geometries of rectangular, symmetric, Hermitian, and alternate matrices in the middle forties of the last century. The elements of such a matrix space are also called *points*, and there is a symmetric and anti-reflexive *adjacency* relation on the point set. The adjacency relation turns the point set of a matrix space into the set of vertices of a graph. The problem to describe all isomorphisms between such graphs has attracted many authors. In other words, one aims at describing all bijections between matrix spaces such that adjacency (graph-theoretic distance 1) is preserved in both directions. Solutions to this problem are usually stated as a *fundamental theorem* for a geometry of matrices. See the book of Z.-X. Wan [?] for a wealth of results and references.

All graphs, which stem from the matrix spaces mentioned above, have finite diameter. Several recent papers are concerned with a description of all bijections between matrix spaces which are *diameter preserving* in both directions. The proofs pursue the same pattern: In a first step, a bijection of this kind is shown to preserve adjacency in both directions. Then, in a second step, the appropriate

*Lise Meitner Research Fellow of the Austrian Science Fund (FWF), project M 1023.

†Corresponding author

fundamental theorem is applied to accomplish the task. See [?] and [?]. Similar results about Grassmann spaces and other structures can be found in [?], [?], [?], and [?].

In the present paper we aim at shedding light on this issue by a different approach. It follows the ideas from [?], where adjacency preserving mappings were exhibited for a wide class of point-line geometries rather than those of a specific kind. So, we consider a *class of graphs* subject to five conditions (A1)–(A5), one of them ensuring finiteness of diameters. Theorem 2.2 contains our main result: A surjective mapping φ between graphs Γ and Γ' from this class is an isomorphism provided that any two points of Γ are at a distance equal to the diameter of Γ if, and only if, their images are at a distance equal to the diameter of Γ' . The backbone of the proof is contained in Lemma 2.1, which is about graphs satisfying (A1)–(A4). It contains a sufficient condition for two points of such a graph to be adjacent. This condition is in terms of the diameter alone (cf. formula (1)), and it appears also in the articles mentioned before. The remaining condition (A5) just assures that any two adjacent points admit a description as in this lemma.

In this way we set up a very general framework which can then be applied to several geometries of matrices. We verify conditions (A1)–(A5) for the geometry of rectangular matrices over an division ring with more than two elements, the geometry of Hermitian matrices over a division ring with involution satisfying some extra conditions, and the projective geometry of rectangular matrices over an arbitrary division ring. Consequently, Theorem 2.2 is applicable to all these geometries. This improves results from [?], [?], and [?] by removing unnecessary assumptions. At the end of Subsection 3.2 we present several examples, for which some of the conditions (A1)–(A5) are violated. In particular, it is shown that a diameter preserving surjection need not be an isomorphism for spaces of symmetric $n \times n$ matrices, n even, over a field of characteristic 2.

We are convinced that there are many more geometries, which allow an interpretation as a graph with properties (A1)–(A5). Thus, our main result should also find other applications in the future.

On the other hand, a condition in the spirit of our Lemma 2.1 was also used in situations which are beyond our approach. See [?], where the points of a graph are defined to be certain subspaces of a vector space with infinite dimension, and [?], where all bounded linear operators of a complex Hilbert space with infinite dimension are considered as points of a graph. Any of the graphs arising in one of these ways has infinite diameter. Nevertheless it is possible to characterise its adjacency relation in terms of another, extrinsically given, binary relation. This relation is the complementarity of two subspaces in [?] and the invertibility of the difference of two operators in [?].

2 The main result

Let Γ be a (finite or infinite) graph. Note that all our graphs are undirected, without loops and with at least one vertex. The set of vertices of Γ will be

denoted by \mathcal{P} . In a more geometric language, vertices will also be called *points*. As usual, we say that $x, y \in \mathcal{P}$ are *adjacent* if $\{x, y\}$ is an edge. The *distance* of two points $x, y \in \mathcal{P}$ is written as $d(x, y)$. Thus x, y are adjacent precisely when $d(x, y) = 1$.

From now on, we focus our attention on graphs Γ satisfying the following conditions:

(A1) Γ is connected and its diameter $\text{diam } \Gamma$ is finite.

(A2) For any points $x, y \in \mathcal{P}$ there is a point $z \in \mathcal{P}$ with

$$d(x, z) = d(x, y) + d(y, z) = \text{diam } \Gamma.$$

(A3) For any points $x, y, z \in \mathcal{P}$ with $d(x, z) = d(y, z) = 1$ and $d(x, y) = 2$ there is a point w satisfying

$$d(x, w) = d(y, w) = 1 \text{ and } d(z, w) = 2.$$

(A4) For any points $x, y, z \in \mathcal{P}$ with $x \neq y$ and $d(x, z) = d(y, z) = \text{diam } \Gamma$ there is a point w with

$$d(z, w) = 1, \quad d(x, w) = \text{diam } \Gamma - 1, \quad \text{and} \quad d(y, w) = \text{diam } \Gamma.$$

(A5) For any adjacent points $a, b \in \mathcal{P}$ there exists a point $p \in \mathcal{P} \setminus \{a, b\}$ such that for all $x \in \mathcal{P}$ the following holds:

$$d(x, p) = \text{diam } \Gamma \Rightarrow d(x, a) = \text{diam } \Gamma \vee d(x, b) = \text{diam } \Gamma.$$

Let us shortly comment on these conditions: (A1) is merely a technical assumption which is needed for all that follows. The subsequent conditions are about geodesics of Γ : (A2) says that any geodesic can be extended at each of its endpoints to a geodesic with length $\text{diam } \Gamma$, which is the maximal length any geodesic might have. Condition (A3) ensures that for any two points x, y at distance 2 there are geodesics (x, z, y) and (x, w, y) with $d(z, w) = 2$. It appears also in [?] and [?]. Similarly, (A4) guarantees for distinct points x, y the existence of a geodesic (x, \dots, w, z) subject to the specified property of the penultimate point w . Finally, we have our crucial condition (A5): It states for any two adjacent points the existence of a third point with certain properties.

We refer to Section 3 for infinite series of graphs which satisfy (A1)–(A5). Graphs which satisfy (A1)–(A3), but only one of (A4) and (A5) are presented in Example 3.7 and Example 3.8.

Our first result contains a sufficient condition for two points to be adjacent. Observe that we do not assume condition (A5) here.

Lemma 2.1. *Given a graph Γ which satisfies conditions (A1)–(A4) let $n := \text{diam } \Gamma$. Suppose that $a, b \in \mathcal{P}$ are distinct points with the following property:*

$$\exists p \in \mathcal{P} \setminus \{a, b\} \forall x \in \mathcal{P} : d(x, p) = n \Rightarrow d(x, a) = n \vee d(x, b) = n. \quad (1)$$

Then a and b are adjacent.

Proof. Let $k := d(a, p)$. First we show $k = 1$. By condition (A2), there is a point $x \in \mathcal{P}$ with

$$n = d(x, p) = d(x, a) + d(a, p).$$

Thus $d(x, a) = n - k < n$. We read off from (1) that $d(x, b) = n$. Now condition (A4) implies the existence of a point $y \in \mathcal{P}$ with

$$d(x, y) = 1, \quad d(y, b) = n - 1, \quad d(y, p) = n.$$

So (1) yields $d(y, a) = n$. Finally,

$$n = d(y, a) \leq d(y, x) + d(x, a) = 1 + n - k$$

implies $k = 1$, as required. Since property (1) is symmetric in a and b , we also have $d(b, p) = 1$.

Now we prove $d(a, b) = 1$. Suppose to the contrary $d(a, b) \neq 1$. From $d(a, b) \leq d(a, p) + d(p, b) = 2$, we obtain $d(a, b) = 2$. Condition (A3) yields the existence of a point $w \in \mathcal{P}$ with

$$d(a, w) = d(b, w) = 1 \quad \text{and} \quad d(p, w) = 2.$$

By (A2), there is a point $z \in \mathcal{P}$ with

$$n = d(z, p) = d(z, w) + d(w, p).$$

Therefore $d(z, w) = n - 2$. Furthermore, $d(a, z) \leq d(a, w) + d(w, z) = n - 1$ and $d(b, z) \leq d(b, w) + d(w, z) = n - 1$, a contradiction to property (1). \square

We are now in a position to prove our main theorem.

Theorem 2.2. *Let Γ and Γ' be two graphs satisfying the above conditions (A1)–(A5). If $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$ is a surjection which satisfies*

$$d(x, y) = \text{diam } \Gamma \iff d(x^\varphi, y^\varphi) = \text{diam } \Gamma' \quad \text{for all } x, y \in \mathcal{P}, \quad (2)$$

then φ is an isomorphism of graphs. Consequently, $\text{diam } \Gamma = \text{diam } \Gamma'$.

Proof. We start by showing that φ is injective. There are two cases as follows.

$\text{diam } \Gamma' = 0$: Choose any $x \in \mathcal{P}$. From $0 = d(x^\varphi, x^\varphi) = \text{diam } \Gamma'$ follows $0 = d(x, x) = \text{diam } \Gamma$. This implies $|\mathcal{P}| = 1$, whence φ is injective.

$\text{diam } \Gamma' \geq 1$: Let $x, y \in \mathcal{P}$ be distinct. If $d(x, y) = \text{diam } \Gamma$ then $d(x^\varphi, y^\varphi) = \text{diam } \Gamma' \geq 1$ so that $x^\varphi \neq y^\varphi$. Now suppose that $d(x, y) < \text{diam } \Gamma$. Then, by (A2) and $x \neq y$, there exists a point $z \in \mathcal{P}$ for which

$$d(x, z) = d(x, y) + d(y, z) = \text{diam } \Gamma \neq d(y, z).$$

Hence $d(x^\varphi, z^\varphi) = \text{diam } \Gamma' \neq d(y^\varphi, z^\varphi)$ which shows $x^\varphi \neq y^\varphi$.

By the above, we are given a bijection $\varphi : \mathcal{P} \rightarrow \mathcal{P}'$. We infer from Lemma 2.1 and (A5), that φ preserves adjacency of points in both directions. Hence it is an isomorphism of graphs. \square

3 Applications

3.1 Geometry of rectangular matrices

Let \mathcal{D} be a division ring, $|\mathcal{D}| \neq 2$, and let $m, n \geq 2$ be integers. The space of rectangular matrices is based upon set $M_{m \times n}(\mathcal{D})$ of $m \times n$ matrices with entries in \mathcal{D} . Two matrices $A, B \in M_{m \times n}(\mathcal{D})$ are defined to be *adjacent* if

$$\text{rank}(A - B) = 1.$$

Here the term “rank of a matrix” is always understood to be the *left row rank*, i. e., it equals the dimension of the subspace spanned by the row vectors of the matrix in the left vector space \mathcal{D}^n . It is well known that the left row rank and the *right column rank* coincide for any matrix. As adjacency is an anti-reflexive and symmetric relation on $M_{m \times n}(\mathcal{D})$, it can be viewed as the adjacency relation of a graph with point set $M_{m \times n}(\mathcal{D})$. It was proved in [?, Proposition 3.5] that

$$d(A, B) = \text{rank}(A - B) \text{ for all } A, B \in M_{m \times n}(\mathcal{D}). \quad (3)$$

We recall that the group $GM_{m \times n}(\mathcal{D})$ of transformations

$$M_{m \times n}(\mathcal{D}) \rightarrow M_{m \times n}(\mathcal{D}) : X \mapsto PXQ + R, \quad (4)$$

where $P \in \text{GL}_m(\mathcal{D})$, $Q \in \text{GL}_n(\mathcal{D})$, and $R \in M_{m \times n}(\mathcal{D})$, is a subgroup of the automorphism group of the graph on $M_{m \times n}(\mathcal{D})$.

It was shown in [?, Corollary 3.10] that any two adjacent points $X, Y \in M_{m \times n}(\mathcal{D})$ belong to precisely two maximal cliques. Their intersection is defined to be the *line* joining X and Y ; see [?, Corollary 3.13]. Moreover, the following holds by [?, Lemma 2.2]: Given a point $P \in M_{m \times n}(\mathcal{D})$ and a line then either (i) all points of this line are at the same distance from P or (ii) there is an integer $k \geq 1$ such that precisely one point of this line is at distance $k - 1$ from P , and all other points of this line are at distance k from P . We shall use this result below.

Lemma 3.1. *The graph on $M_{m \times n}(\mathcal{D})$ satisfies conditions (A1)–(A5).*

Proof. We denote by $E_{jk} \in M_{m \times n}(\mathcal{D})$ the matrix whose (j, k) entry equals 1, whereas all other entries are 0. All unordered pairs of matrices with a fixed distance k are in one orbit under the action of the group $GM_{m \times n}(\mathcal{D})$. When exhibiting such a pair we may therefore assume without loss of generality the two matrices to be 0 and $E_{11} + E_{22} + \cdots + E_{kk}$.

First, we restrict ourselves to the case $n \geq m$.

Ad (A1): This is immediate from (3).

Ad (A2): Let $X = 0$ and $Y = \sum_{i=1}^k E_{ii}$. Then $Z := \sum_{j=1}^n E_{jj}$ has the required properties.

Ad (A3): Let $Y = E_{11}$, $Z = 0$, and X be given, where $d(X, Z) = \text{rank}(X) = 1$ and $d(X, Y) = 2$. The line joining Y and Z equals $\{uE_{11} \mid u \in \mathcal{D}\}$. The points $Y, Z, -Y$ are on this line. By the preceding remark, all points of this

line, except for Z , are at distance 2 from X . In particular, $d(X, -Y) = 2$. Now define $W := X + Y$. Then $d(W, X) = \text{rank}(Y) = 1$, $d(W, Y) = \text{rank}(X) = 1$ and $d(W, Z) = \text{rank}(X - (-Y)) = d(X, -Y) = 2$.

Ad (A4): Let $X \neq Y$ and $Z = 0$, whence $\text{rank}(X) = \text{rank}(Y) = m$. With x_1, x_2, \dots, x_m and y_1, y_2, \dots, y_m denoting the row vectors of X and Y , respectively, we claim that there exists such an $i \in \{1, 2, \dots, m\}$ that in \mathcal{D}^n the $(m-1)$ -dimensional affine subspaces

$$\begin{aligned} U_{X,i} &:= x_i + \text{span}(x_1, x_2, \dots, \hat{x}_i, \dots, x_m), \\ U_{Y,i} &:= y_i + \text{span}(y_1, y_2, \dots, \hat{y}_i, \dots, y_m) \end{aligned}$$

are distinct. (The notation \hat{x}_i means that this vector is omitted.) Assume to the contrary that this would not be the case. Then, for any fixed index $j \in \{1, 2, \dots, m\}$, we would obtain that

$$x_j \in \text{span}(x_1, x_2, \dots, \hat{x}_k, \dots, x_m) = \text{span}(y_1, y_2, \dots, \hat{y}_k, \dots, y_m)$$

for all $k \neq j$, whence $x_j \in \text{span}(y_j)$ due to the linear independence of the row vectors of Y . Furthermore, $U_{X,j} = U_{Y,j}$ would give $x_j = y_j$. Since j was chosen arbitrarily, we would obtain $X = Y$, a contradiction.

So, we may choose a vector $w \in U_{X,i} \setminus U_{Y,i}$. Define a matrix $W \in M_{m \times n}(\mathcal{D})$ as follows: Its i th row is equal to w , all other rows are 0. Then $\text{rank}(W) = 1$, $\text{rank}(X - W) = m - 1$, and $\text{rank}(Y - W) = m$, as required.

Ad (A5): It suffices to consider the case $A = 0$ and $B = E_{11}$. By $|\mathcal{D}| \neq 2$, the line $\{uE_{11} \mid u \in \mathcal{D}\}$ contains a point $P \neq A, B$. Let $X \in M_{m \times n}(\mathcal{D})$ be any point with $d(X, P) = m$. By the remarks preceding Lemma 3.1 and due to the fact that points with distance $m+1$ do not exist, at most one of A and B is at distance $m-1$ from X .

The case $n \leq m$ can be shown similarly by considering columns of matrices as vectors of a right vector space over \mathcal{D} . \square

By combining Theorem 2.2 and Lemma 3.1 we obtain:

Theorem 3.2. *Let $\mathcal{D}, \mathcal{D}'$ be division rings with $|\mathcal{D}|, |\mathcal{D}'| \neq 2$. Let m, n, p, q be integers ≥ 2 . If $\varphi : M_{m \times n}(\mathcal{D}) \rightarrow M_{p \times q}(\mathcal{D}')$ is a surjection which satisfies*

$$\begin{aligned} \text{rank}(A - B) = \min\{m, n\} &\Leftrightarrow \text{rank}(A^\varphi - B^\varphi) = \min\{p, q\} \\ &\text{for all } A, B \in M_{m \times n}(\mathcal{D}), \end{aligned}$$

then φ is bijective. Both φ and φ^{-1} preserve adjacency of matrices. Moreover, $\min\{m, n\} = \min\{p, q\}$.

The fundamental theorem of the geometry of rectangular matrices [?, Theorem 3.4] can be used to explicitly describe a mapping φ as in the theorem. As a further consequence, the existence of φ implies that \mathcal{D} and \mathcal{D}' are isomorphic or anti-isomorphic division rings, and that $\{m, n\} = \{p, q\}$.

3.2 Geometry of Hermitian and symmetric matrices

Let \mathcal{D} be a division ring which possesses an involution, i. e. an anti-automorphism of \mathcal{D} whose square equals the identity map id of \mathcal{D} . Throughout this subsection, we choose one involution, say $\bar{}$, of \mathcal{D} . Also, we assume that the following restrictions are satisfied:

- (R1) The set \mathcal{F} of fixed elements of $\bar{}$ has more than three elements in common with the centre $Z(\mathcal{D})$ of \mathcal{D} .
- (R2) When $\bar{}$ is the identity map, whence $\mathcal{D} = \mathcal{F}$ is a field, then assume that \mathcal{F} does not have characteristic 2 (in symbols: $\text{char}(\mathcal{F}) \neq 2$).

Let $\mathcal{H}_n(\mathcal{D})$ denote the space of Hermitian $n \times n$ matrices over \mathcal{D} (with respect to $\bar{}$), where $n \geq 2$. If $\bar{}$ is the identity map, then $\mathcal{H}_n(\mathcal{D}) =: \mathcal{S}_n(\mathcal{F})$ is the space of symmetric $n \times n$ matrices over \mathcal{F} .

We call any Hermitian matrix in $\mathcal{H}_n(\mathcal{D})$ a *point* and adopt the *adjacency relation* from 3.1, i. e., $A, B \in \mathcal{H}_n(\mathcal{D})$ are adjacent precisely when $\text{rank}(A - B) = 1$. This turns $\mathcal{H}_n(\mathcal{D})$ into a graph. We recall that the group $G\mathcal{H}_n(\mathcal{D})$ of transformations

$$\mathcal{H}_n(\mathcal{D}) \rightarrow \mathcal{H}_n(\mathcal{D}) : X \mapsto PX\bar{P}^t + H, \quad (5)$$

where $P \in \text{GL}_n(\mathcal{D})$ and $H \in \mathcal{H}_n(\mathcal{D})$, is a subgroup of the automorphism group of the graph on $\mathcal{H}_n(\mathcal{D})$.

For any two matrices $A, B \in \mathcal{H}_n(\mathcal{D})$ the distance $d(A, B)$ in the graph on $\mathcal{H}_n(\mathcal{D})$ equals $\text{rank}(A - B)$. This can be shown, *mutatis mutandis*, as in [?, Proposition 5.5], because (R2) guarantees that any Hermitian matrix is cogredient to a matrix of the form $\sum_{i=1}^n a_i E_{ii}$ with $a_i \in \mathcal{F}$. See, for example, [?, p. 15].

Lemma 3.3. *Let $A \in \mathcal{H}_n(\mathcal{D})$ be a matrix with $\text{rank}(A) = k + 1$. A matrix $B \in \mathcal{H}_n(\mathcal{D})$ has rank 1 and $\text{rank}(A - B) = k$ if, and only if, there exists an $x \in \mathcal{D}^n$ with $xA\bar{x}^t \neq 0$ and*

$$B = (\overline{xA})^t (xA\bar{x}^t)^{-1} (xA).$$

Proof. This is a slight generalisation of Lemma 2.2 in [?], since we do *not* assume $\mathcal{F} \subseteq Z(\mathcal{D})$. However, the proof given there can be carried over to our more general settings in a straightforward way. On the one hand, all scalars in \mathcal{F} (like $(xA\bar{x}^t)^{-1}$ in the definition of B from above) have to be written *between* a matrix and its Hermitian transpose rather than on the left hand side (as in [?]). Also, one has to take into account what we already noticed before: In the presence of restriction (R2), any Hermitian matrix is cogredient to a diagonal matrix (with entries in \mathcal{F}) irrespective of whether \mathcal{F} is in the centre of \mathcal{D} or not. \square

Lemma 3.4. *Let $A, B \in \mathcal{H}_n(\mathcal{D})$ be non-zero, and suppose that there exists $P \in \text{GL}_n(\mathcal{D})$ such that*

$$PA\bar{P}^t = \text{diag}(a_1, a_2, \dots, a_k, 0, \dots, 0) \quad \text{and} \quad PB\bar{P}^t = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $k = \text{rank}(A)$ and B_1 denotes a Hermitian matrix of size $\leq k$. Then there is a vector $x \in \mathcal{D}^n$ such that

$$xA\bar{x}^t \neq 0 \quad \text{and} \quad xB\bar{x}^t \neq 0.$$

Proof. Without loss of generality, let $k = n$, whence $A = \text{diag}(a_1, a_2, \dots, a_n)$, $a_i \in \mathcal{F} \setminus \{0\}$, and $B = (b_{ij}) \neq 0$.

Case 1. $b_{ii} \neq 0$ for some i . Then e_i , viz. the i th vector of the canonical basis of \mathcal{D}^n , satisfies

$$e_i A \bar{e}_i^t = a_i \neq 0 \quad \text{and} \quad e_i B \bar{e}_i^t = b_{ii} \neq 0.$$

Case 2. $b_{ii} = 0$ for all i . Since $B \neq 0$, there exist i, j with $1 \leq i, j \leq n$ and $i \neq j$ such that $b_{ij} \neq 0$. Without loss of generality, we assume $b_{12} \neq 0$. Let $x = (x_1, 1, 0, \dots, 0)$, then $xA\bar{x}^t = x_1 a_1 \bar{x}_1 + a_2$ and $xB\bar{x}^t = x_1 b_{12} + \overline{x_1 b_{12}}$, so it is enough to find $x_1 \in \mathcal{D}$ such that

$$x_1 a_1 \bar{x}_1 \neq -a_2 \quad \text{and} \quad x_1 b_{12} \neq -\overline{x_1 b_{12}}.$$

As $|\mathcal{F} \cap Z(\mathcal{D})| > 3$, there exists $\lambda \in (\mathcal{F} \cap Z(\mathcal{D})) \setminus \{0\}$ with $\lambda^2 \neq 1$. Note that $\mathcal{D} = \{\xi \in \mathcal{D} \mid \xi = -\bar{\xi}\}$ would imply $(\bar{}) = \text{id}$ and $\text{char } \mathcal{D} = 2$, which contradicts (R2). So, there is $x'_1 \in \mathcal{D}$ with $x'_1 b_{12} \neq -\overline{x'_1 b_{12}}$. Define $x_1 := x'_1$ if $x_1 a_1 x'_1 \neq -a_2$, and $x_1 := \lambda x'_1$ if $x_1 a_1 x'_1 = -a_2$. \square

Lemma 3.5. *The graph on $\mathcal{H}_n(\mathcal{D})$ satisfies conditions (A1)–(A5).*

Proof. When exhibiting two Hermitian matrices with distance k , we may assume, by virtue of the action of $G\mathcal{H}_n(\mathcal{D})$, the matrices to be 0 and $a_1 E_{11} + a_2 E_{22} + \dots + a_k E_{kk}$ with $a_1, a_2, \dots, a_k \in \mathcal{F}$. Taking into account the previous remark, the proof for (A1), (A2), (A3), and (A5) can be carried over almost unchanged from the proof of Lemma 3.1. Only certain scalars have to be chosen from $\mathcal{F} \cap Z(\mathcal{D})$ rather than \mathcal{D} .

Our proof of (A4) is different though: Let $X \neq Y$ and Z be matrices in $\mathcal{H}_n(\mathcal{D})$ with $d(X, Z) = n$ and $d(Y, Z) = n$. Without loss of generality, we assume $Z = 0$ and $\text{rank}(X) = \text{rank}(Y) = n$. From Lemma 3.4, applied to $A := X$ and $B := X - XY^{-1}X \neq 0$, there exists a vector $v \in \mathcal{D}^n$ such that

$$vX\bar{v}^t \neq 0 \quad \text{and} \quad vX\bar{v}^t - v(XY^{-1}X)\bar{v}^t \neq 0.$$

We define

$$W := (\overline{vX})^t (vX\bar{v}^t)^{-1} (vX) \in \mathcal{H}_n(\mathcal{D}). \quad (6)$$

Then $d(Z, W) = 1$ and $d(Y, W) \geq n - 1$ are obvious, whereas Lemma 3.4 shows $d(X, W) = n - 1$. Let us suppose $d(Y, W) = n - 1$. By Lemma 3.3, there exists a vector $u \in \mathcal{D}^n$ such that

$$W = (\overline{uY})^t (uY\bar{u}^t)^{-1} (uY). \quad (7)$$

We infer from (7) and (6) that uY and vX are left-proportional by a non-zero factor in \mathcal{D} . Since u is determined up to a non-zero factor in \mathcal{D} only, we may therefore even suppose $uY = vX$. Comparing (7) with (6) yields $vX\bar{v}^t = uY\bar{u}^t$. This implies that $vX\bar{v}^t - v(XY^{-1}X)\bar{v}^t = 0$, a contradiction. So we must have $d(Y, W) = n$. \square

Theorem 3.6. *Let $\mathcal{D}, \mathcal{D}'$ be division rings which possess involutions $\bar{}$ and $\bar{}'$, respectively, subject to the restrictions (R1) and (R2). Let n, n' be integers ≥ 2 . If $\varphi : \mathcal{H}_n(\mathcal{D}) \rightarrow \mathcal{H}_{n'}(\mathcal{D}')$ is a surjection which satisfies*

$$\text{rank}(A - B) = n \iff \text{rank}(A^\varphi - B^\varphi) = n' \text{ for all } A, B \in \mathcal{H}_n(\mathcal{D}),$$

then φ is bijective. Both φ and φ^{-1} preserve adjacency of Hermitian matrices. Moreover, $n = n'$.

A prospective *fundamental theorem of the geometry of Hermitian matrices* should describe all bijections $\mathcal{H}_n(\mathcal{D}) \rightarrow \mathcal{H}_{n'}(\mathcal{D}')$ which preserve adjacency in both directions. However, such a fundamental theorem seems to be known only under additional assumptions on the division rings, their involutions, and/or the numbers n, n' . We refer to [?], [?], [?], and [?, Chapter 6] for further details. Each of these results can be used to (i) explicitly describe a mapping φ as in the theorem and (ii) to derive from the existence of φ that \mathcal{D} and \mathcal{D}' are isomorphic division rings.

We close this subsection with some examples in which one or even both of the restrictions (R1) and (R2) dropped.

Example 3.7. Let \mathcal{F}_3 be the field with three elements. We exhibit the space of symmetric 2×2 matrices over \mathcal{F}_3 . The graph on $\mathcal{S}_2(\mathcal{F}_3)$ has 27 points and diameter 2. It is easy to verify conditions (A1), (A2), (A3), and (A5) as before.

In what follows we establish that (A4) is not satisfied. Figure 1 depicts five points of the graph on $\mathcal{S}_2(\mathcal{F}_3)$ and all edges between them. It is straightforward

$$\begin{array}{ccccc} X = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} & & U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = Z \\ & \searrow & & \nearrow & \\ Y = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} & & V = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} & & \end{array}$$

Figure 1: A counterexample for (A4) and Lemma 2.1

to show that (X, U, Z) and (X, V, Z) are the only two geodesics from X to Z . However, both U and V are neighbours of $Y \neq X$, whence we cannot find a matrix W to satisfy (A4).

Furthermore, property (1) holds for $A := X$, $B := Z$, and $P := Y$. Indeed, U and V are the only points of $\mathcal{S}_2(\mathcal{F}_3)$ which are adjacent to A and B , but none of them is at distance 2 from P . Yet, in contrast to the assertion of Lemma 2.1, the points A and B are not adjacent.

Nevertheless, any mapping $\varphi : \mathcal{S}_2(\mathcal{F}_3) \rightarrow \mathcal{S}_2(\mathcal{F}_3)$ as in Theorem 3.6 is an automorphism of the graph on $\mathcal{S}_2(\mathcal{F}_3)$, a fact which is immediate from the following observation: Given a mapping $\varphi : \mathcal{P} \rightarrow \mathcal{P}$ as in Theorem 2.2, where $\Gamma = \Gamma'$ is a finite graph with diameter $\text{diam } \Gamma = 2$, the surjectivity of φ implies its being a bijection. Furthermore, since distance 2 is preserved under φ and φ^{-1} , so is distance 1. Hence φ is an automorphism.

Example 3.8. Let \mathcal{F}_2 be the field with two elements. We exhibit the space of symmetric 2×2 matrices over \mathcal{F}_2 . The graph on $\mathcal{S}_2(\mathcal{F}_2)$ has 8 points and diameter 3, an illustration is given in Figure 2. It is straightforward to show that conditions (A1), (A2), (A3), and (A4) are satisfied, whereas (A5) does not hold.

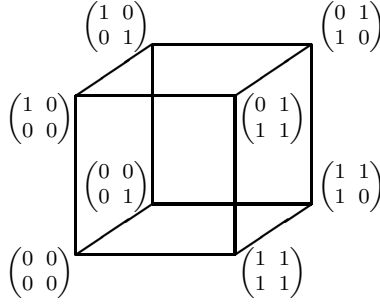


Figure 2: A counterexample for (A5)

Another way of seeing that the graph on $\mathcal{S}_2(\mathcal{F}_2)$ cannot satisfy all conditions (A1)–(A5) is as follows. Suppose that Γ is a graph with diameter $\text{diam } \Gamma \geq 3$ such that there exist points $a, a^* \in \mathcal{P}$ with $d(a, a^*) = \text{diam } \Gamma$ and $d(a, x) \neq \text{diam } \Gamma \neq d(a^*, x)$ for all $x \in \mathcal{P} \setminus \{a, a^*\}$. Let $\varphi : \mathcal{P} \rightarrow \mathcal{P}$ be the bijection which interchanges a with a^* and leaves invariant all other points. This φ preserves pairs of points with distance $\text{diam } \Gamma$ in both directions. But, due to $\text{diam } \Gamma \geq 3$, the bijection φ cannot be an automorphism of Γ . Clearly, the graph on $\mathcal{S}_2(\mathcal{F}_2)$ is of this kind.

Example 3.9. The space $\mathcal{S}_2(\mathcal{F}_2)$ from Example 3.8 is just a particular case of the following, more general situation. Let \mathcal{F} be any field of characteristic 2, and let $n \geq 2$ be an even integer. By [?, Proposition 5.5], the diameter of the graph on the space $\mathcal{S}_n(\mathcal{F})$ equals $n + 1 \geq 3$. Moreover, two matrices $A, B \in \mathcal{S}_n(\mathcal{F})$ satisfy $d(A, B) = n + 1$ if, and only if, $A - B$ is an alternate matrix with rank n . Consequently, $d(A, B) = n + 1$ implies that either *both A and B are alternate* or *both A and B are non-alternate*. Now it is easy to establish the existence of a bijection $\varphi : \mathcal{S}_n(\mathcal{F}) \rightarrow \mathcal{S}_n(\mathcal{F})$ which preserves pairs of matrices at distance $n + 1$ in both directions without being an isomorphism. Choose any alternate matrix $K \in \mathcal{S}_n(\mathcal{F})$ with $K \neq 0$. Given $X \in \mathcal{S}_n(\mathcal{F})$ we define

$$X^\varphi := X + K \text{ if } X \text{ is alternate, and } X^\varphi := X \text{ otherwise.}$$

As the restriction of φ to the set of alternate matrices is a transformation as in (5), φ preserves matrix pairs with distance $n + 1$. We have $d(E_{11}, 0) = 1$ and

$$d(0, E_{11}) + d(E_{11}, K) \geq d(0, K) = \text{rank}(K) + 1 \geq 3.$$

Hence $d(E_{11}^\varphi, 0^\varphi) = d(E_{11}, K) \geq 2$.

3.3 Projective geometry of rectangular matrices—the Grassmann space

Let \mathcal{D} be a division ring. The projective space of rectangular matrices $M_{m \times n}(\mathcal{D})$, $m, n \geq 2$, is the Grassmann space $G(m, m+n; \mathcal{D})$ over \mathcal{D} ; its *points* are the m -dimensional subspaces of the $(m+n)$ -dimensional left vector space over \mathcal{D} . We refer to [?, Section 3.6] for its relationship with $M_{m \times n}(\mathcal{D})$. Two points $W_1, W_2 \in G(m, m+n; \mathcal{D})$ are called *adjacent* if $W_1 \cap W_2$ is $(m-1)$ -dimensional. As before, we consider $G(m, m+n; \mathcal{D})$ as a graph based on the adjacency relation. The distance between two points W_1 and W_2 is

$$d(W_1, W_2) = m - \dim(W_1 \cap W_2).$$

The graph on the Grassmann space $G(m, m+n; \mathcal{D})$ has diameter $\min\{m, n\}$.

Using dimension arguments, conditions (A1), (A2), (A3), and (A5) can be proved easily. We sketch the proof of (A4) for the case $m \leq n$. Given m -dimensional subspaces X, Y, Z with $X \neq Y$ and $d(X, Z) = d(Y, Z) = m$ there exists a vector $a \in X \setminus Y$. Choose an $(m-1)$ -dimensional subspace $S \subset Z$ such that $S \cap (\text{span}(a, Y) \cap Z) = \{0\}$. Then $W := \text{span}(a, S)$ has the required properties.

Due to the presence of *points at infinity* there is no need to exclude the field with two elements in the following theorem.

Theorem 3.10. *Let $\mathcal{D}, \mathcal{D}'$ be division rings. Let m, n, p, q be integers ≥ 2 . If $\varphi : G(m, m+n; \mathcal{D}) \rightarrow G(p, p+q; \mathcal{D}')$ is a surjection which satisfies*

$$d(A, B) = \min\{m, n\} \Leftrightarrow d(A^\varphi, B^\varphi) = \min\{p, q\} \\ \text{for all } A, B \in G(m, m+n; \mathcal{D}),$$

then φ is bijective. Both φ and φ^{-1} preserve adjacency of subspaces. Moreover, $\min\{m, n\} = \min\{p, q\}$.

The fundamental theorem of the projective geometry of rectangular matrices [?, Theorem 3.52] can be used to explicitly describe a mapping φ as in the theorem. As a further consequence, the existence of φ implies that \mathcal{D} and \mathcal{D}' are isomorphic or anti-isomorphic division rings, and that $\{m, n\} = \{p, q\}$.

Wen-ling Huang, Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8–10, A-1040 Wien, Austria.
Department Mathematik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany.
`huang@math.uni-hamburg.de`

Hans Havlicek, Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8–10, A-1040 Wien, Austria.
`havlicek@geometrie.tuwien.ac.at`